OBERSEMINAR: CONDENSED MATHEMATICS

Introduction

Condensed mathematics was introduced by Clausen–Scholze in 2019 as a new foundation for topology that behaves much better in the context of algebraic structures. As such, it provides new and powerful techniques in the study of geometric spaces with "topological" coordinate rings like adic spaces and complex analytic spaces, where a good notion of quasi-coherent sheaves is now possible. But even in classically discrete situations like algebraic geometry, the condensed formalism allows new insights: Most prominently, by adding non-trivial "topological" structures to quasi-coherent sheaves on schemes, one can build a full 6-functor formalism for quasi-coherent sheaves which results in easy proofs of many foundational results on coherent and quasi-coherent sheaves.

The power of condensed mathematics comes from solving the following core problems of classical topological algebra:

(a) The category of topological abelian groups is not an abelian category. For example, let $\mathbb{R}_{\text{disc}}$ denote the (additive) group of real numbers equipped with the discrete topology; then the identity map $\mathbb{R}_{\text{disc}} \to \mathbb{R}$ is a map of topological abelian groups whose kernel and cokernel vanish, but it is not an isomorphism. Having no abelian category of topological abelian groups available has many annoying consequences, most prominently it makes it hard to work with derived structures. For example, if G is a topological group then a short exact sequence of topological G-modules does not produces a long exact sequence of G-cohomology in general.

(b) In the context of geometric objects with topological coordinate rings, a good notion of quasicoherent sheaves is hard to define. One of the problems is that one needs a general notion of a completed base-change $M \widehat{\otimes}_A B$ for a map $A \to B$ of topological rings and a topological A-module M.

Let us first discuss how problem (a) is solved. Condensed mathematics redefines the notion of topological spaces (and thereby of topological abelian groups) by roughly allowing non-trivial "topologies" on trivial underlying sets. In the example of the map $f\colon \mathbb{R}_{\text{disc}} \to \mathbb{R}$ this results in a non-trivial $\operatorname{coker} f$ (even though its underlying set is still 0). To make this abstract idea into a concrete definition, the basic observation of condensed mathematics is that all "nice" topological spaces X are determined by the continuous maps $S \to X$ they allow from all compact Hausdorff spaces S. For example, let $S = \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$. Then a continuous map $\mathbb{N}_\infty \to X$ is the same as a converging sequence of X, hence the maps $\mathbb{N}_\infty \to X$ determine the convergence of sequences in X.

Motivated by the observations in the previous paragraph, we define a *condensed set* X to be a sheaf of sets on the site of compact Hausdorff spaces (where coverings are finite jointly surjective families of continuous maps), where we view its value $X(S)$ on a compact Hausdorff space as the "continuous maps" $S \to X$. This definition gets simplified by noting that the site of compact Hausdorff spaces has a basis given by profinite sets (i.e. limits of finite sets), which also removes the dependence on any classical topology in the definition of condensed sets:

Definition 1. A *condensed set/abelian group/ring* is a sheaf of sets/abelian groups/rings on the site of profinite sets, where coverings are finite families of jointly surjective maps.

Like sheaves on any site, it is now completely formal that condensed abelian groups form a nice abelian category Cond(Ab). Moreover, the site of profinite sets is locally weakly contractible which implies that Cond(Ab) is generated by compact projective objects $\mathbb{Z}[S]$ for so-called *extremally disconnected* sets S – here $\mathbb{Z}[S]$ is the free abelian sheaf associated to S. In particular there is a good notion of the derived category $\mathcal{D}(\text{Cond}(\text{Ab}))$ which gives us all the tools of homological algebra at hand. One checks that condensed abelian groups and condensed sets capture the classical cohomology of topological spaces and

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that higher Ext-groups of locally compact abelian groups are extremely well-behaved and coincide with previous ad-hoc definitions. This solves problem (a) above.

We now discuss how problem (b) is solved, i.e. the need for some "completed tensor product". While condensed abelian groups come with a tensor product operation, this just operates as the usual tensor product on the underlying abstract abelian groups and is therefore not enough to solve (b). The solution to this problem is by adding a "completeness" notion to the condensed rings we work with, in the following way:

Definition 2. An *analytic ring* is a pair $\mathcal{A} = (\underline{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$ where $\underline{\mathcal{A}}$ is a (derived) condensed ring and $\mathcal{M}_{\mathcal{A}}$ is a functor associating to every extremally disconnected set S some $\underline{\mathcal{A}}$-module $\mathcal{A}[S]$ such that several conditions are satisfied (see Definition 6.1 below).

Given an analytic ring $\mathcal{A}$, one can then define the category of (derived) $\mathcal{A}$-modules $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\underline{\mathcal{A}})$ to be the one generated by $\mathcal{A}[S]$ for extremally disconnected sets S. There is a "completion" functor $- \otimes_{\underline{\mathcal{A}}} \mathcal{A} \colon \mathcal{D}(\underline{\mathcal{A}}) \to \mathcal{D}(\mathcal{A})$ which commutes with colimits and maps $\underline{\mathcal{A}}[S] \mapsto \mathcal{A}[S]$. We can thus define the "$\mathcal{A}$-complete" tensor product as

$$M \otimes_{\mathcal{A}} N := (M \otimes_{\underline{\mathcal{A}}} N) \otimes_{\underline{\mathcal{A}}} \mathcal{A}.$$

The usefulness of this notion comes from the plentiness of examples. The most basic example is the following:

Theorem 3. *For every profinite set $S = \varprojlim_i S_i$ let $\mathbb{Z}_\square[S] := \varprojlim_i \mathbb{Z}[S_i]$. Then the pair $\mathbb{Z}_\square = (\mathbb{Z}, S \mapsto \mathbb{Z}_\square[S])$ is an analytic ring.*

A $\mathbb{Z}_\square$-module is often called a *solid abelian group*. One can show identities like $\mathbb{Z}_p \otimes_{\mathbb{Z}_\square} \mathbb{Z}[[x]] = \mathbb{Z}_p[[x]]$, which indicate that the solid tensor product does indeed provide some form of "completion". Note however that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}_\square = 0$, i.e. the solid theory does not allow us to talk about $\mathbb{R}$-vector spaces. There are analytic rings that work over $\mathbb{R}$, but these will not be the focus of this seminar.

One can generalize the definition of $\mathbb{Z}_\square$ to define an analytic ring $A_\square$ for every discrete ring A. More generally, for every complete Huber pair (A, A^+) one gets an associated analytic ring $(A, A^+)_\square$ and this association is fully faithful. We denote the associated categories of (derived) modules by $\mathcal{D}_\square(A) := \mathcal{D}(A_\square)$ and $\mathcal{D}_\square(A, A^+) := \mathcal{D}((A, A^+)_\square)$. We now get the following result:

Theorem 4. *Let X be a scheme or an analytic adic space. Then the association* $\operatorname{Spec} A \mapsto \mathcal{D}_\square(A)$*, resp.* $\operatorname{Spa}(A, A^+) \mapsto \mathcal{D}_\square(A, A^+)$ *glues along open subsets of X in order to produce an ∞-category $\mathcal{D}_\square(X)$.*

The objects in $\mathcal{D}_\square(X)$ are called the *solid quasicoherent sheaves on X*. Of course one expects this result to be true for every adic space, but this has not been worked out in the literature yet. In the case of analytic adic spaces we finally get a good notion of quasicoherent sheaves. In the case of schemes we get a definition of "complete topological" quasicoherent sheaves, which contain the classical derived category of quasicoherent sheaves.

In the case of schemes, the major advantage of solid quasicoherent sheaves is that they admit a full 6-functor formalism. For classical quasicoherent sheaves this does not work because for a standard open immersion $j \colon \operatorname{Spec} A[1/f] \hookrightarrow \operatorname{Spec} A$ the desired functor $j_! \colon \mathcal{D}(A[1/f]) \to \mathcal{D}(A)$ would need to be a left adjoint of j^*, which requires j^* to commute with limits. However, we have e.g. $A[1/f] \otimes_A \prod_I A \neq \prod_I A[1/f]$ for infinite sets I; the solid formalism solves this problem. The solid 6-functor formalism gives us, for every "nice" (e.g. finite-type) map $f \colon Y \to X$ of schemes, an associated functor $f_! \colon \mathcal{D}_\square(Y) \to \mathcal{D}_\square(X)$ of "pushforward with compact support". If f is proper then $f_! = f_*$. With this tool at hand, many fundamental results about quasicoherent sheaves like Grothendieck duality and coherence of proper pushforward can be reproved in a much simpler fashion: We can now localize these statements (by replacing f_* by $f_!$), which reduces the claim to a simple computation in the case that f is of the form $\operatorname{Spec} A[x] \to \operatorname{Spec} A$.

In the case of analytic adic spaces, a very analogous construction of the 6-functor formalism for solid quasicoherent sheaves is possible and gives us descent of vector bundles, Grothendieck duality and finiteness of coherent cohomology in rigid-analytic geometry. Another application is a proof of the Hirzebruch-Riemann-Roch theorem in the rigid-analytic setting, which was previously only known in complex analytic geometry (and algebraic geometry). The new proof is formal to a large extent and applies in all of these geometric settings.

Notation and Conventions. Starting from Talk 6, we will work completely in the ∞-categorical setting. This has some consequences to the notation: Everything will always be derived, so we will drop any R's or L's to indicate derived functors (e.g. f_* denotes the derived pushforward and $\otimes$ denotes the derived tensor product). Every ring A is assumed to be an animated (i.e. "derived") ring if not stated otherwise, every module is similarly assumed to be derived (i.e. lies in $\mathcal{D}(A)$). We go one step further by additionally allowing all rings and modules to be condensed if not stated otherwise, i.e. by $\mathcal{D}(A)$ we actually denote the derived category of condensed A-modules (and A itself may be a condensed ring). We will say that a ring/module is *static* if it has no derived component (i.e. is concentrated in degree 0) and we will say that it is *discrete* if it has no condensed component (as we will see, discrete rings and modules embed fully faithfully into the condensed world, so this terminology is unambiguous). We sometimes use the word "classical" to mean both static and discrete.

Talks

Talk 1: Condensed Sets. Recall the basic properties of profinite sets and their relation to all compact Hausdorff spaces. Following [8, §1, §2] define condensed sets as sheaves on the site of profinite sets (equivalently on the site compact Hausdorff spaces). Briefly mention how to avoid set-theoretic issues (see [8, §Appendix to Lecture II]). Recall the definition of compactly generated topological spaces and show that they embedd fully faithfully into condensed sets. Introduce injective and surjective maps and show that every condensed set is covered by profinite sets. Using extremally disconnected sets show that the site of profinite sets is locally weakly contractible and use this to describe the quotient $\mathbb{R}/\mathbb{R}_{\mathrm{disc}}$.

Talk 2: Condensed Abelian Groups and Cohomology. Following [8, §2, §3], introduce condensed abelian groups and their general properties (see [8, Theorem 2.2]). Recall singular, sheaf and Čech cohomology of topological spaces with discrete coefficients and relate them to condensed cohomology (see [8, Theorem 3.2]). This motivates viewing condensed cohomology as the general version of cohomology of condensed sets with coefficients in condensed abelian groups; compute the $\mathbb{R}$-cohomology of compact Hausdorff spaces (see [8, Theorem 3.3]).

Talk 3 (1.5 sessions): Locally Compact Abelian Groups. Following [8, §4], define locally compact abelian groups and state their basic properties (see [8, Theorem 4.1]). Use [8, Proposition 4.2] to show that the same properties can be formulated entirely inside the category of condensed abelian groups. Explain the $R\,\underline{\mathrm{Hom}}$-computations in [8, Theorem 4.3] by introducing the Breen-Deligne resolution and the associated spectral sequence. Discuss the fully faithful embedding of discrete abelian groups into condensed abelian groups (on the derived level). Using that $\mathbb{R}/\mathbb{Z}$ is injective as a sheaf on every extremally disconnected space (see [4, Lemma VII.1.7]) deduce that $R\,\underline{\mathrm{Hom}}(M, \mathbb{R}/\mathbb{Z}) = \underline{\mathrm{Hom}}(M, \mathbb{R}/\mathbb{Z})$ for discrete M. Use this to compute all $R\,\underline{\mathrm{Hom}}$'s of locally compact abelian groups and show in particular that all $\underline{\mathrm{Ext}}^i$ vanish for $i \geq 2$.

Talk 4 (1.5 sessions): Solid Abelian Groups. Following [8, §5, §6], define solid abelian groups and prove their main properties. Most prominently, sketch the proof of [8, Theorem 5.8] by using the criterion in [8, Lemma 5.10] and applying all the computations from the previous talks. Then introduce the solid tensor product (see [8, Theorem 6.2]) and provide some basic examples (see [8, Example 6.4]). Also show that the derived solidification of $\mathbb{R}$ is 0 (see [8, Corollary 6.1.(iii)]). If time permits, discuss the Whitehead problem from abstract group theory and how the solid formalism sheds new light on it.

Talk 5: ∞-Categories for the Working Mathematician. Introduce ∞-categories and all the required terminology for the upcoming talks. These include stable ∞-categories, sheaves of ∞-categories, anima, animated rings and symmetric monoidal ∞-categories.

Talk 6: Analytic Rings. Following [8, §7, §8] and [9, §12], introduce (animated) analytic rings and their modules as a generalization of solid abelian groups. Explain that they form nice stable ∞-categories with t-structure and tensor product (see [8, Proposition 7.5] and [9, Proposition 12.4]). Provide some easy examples for analytic rings as in [8, Example 7.3, Proposition 7.9] and some basic constructions like induced analytic rings and topological invariance (see [9, Proposition 12.8 and First Appendix to Lecture

XII]). Introduce morphisms of analytic rings and sketch the construction of colimits in the ∞-category of analytic rings (see [9, Proposition 12.12]).

Talk 7: Discrete Adic Spaces. Construct the analytic ring $A_\square$ associated to every discrete $\mathbb{Z}$-algebra A (following the argument in [8, Theorem 8.1]). Following [6, §2.9, until Lemma 2.9.19], introduce discrete Huber pairs and their associated analytic rings and show their basic properties. Then explain how to glue these analytic rings to define discrete adic spaces and compare them to the classical definition.

Talk 8: The 6-Functor Formalism for Solid Quasicoherent Sheaves I. Explain the abstract notion of a 6-functor formalism (see [8, §11] and [6, §A.5]) and how they help to "localize proper pushforward". Following [6, §2.9, from Lemma 2.9.25], introduce relative compactifications of discrete adic spaces and their relation to +-finite type, separated and proper maps. Further introduce solid quasi-coherent sheaves on discrete adic spaces by gluing them from the affine case and state the existence of the 6-functor formalism (see [6, Proposition 2.9.21]). Reduce the proof of this result to the claim that pullback along open immersions has a left adjoint satisfying Poincaré duality.

Talk 9: The 6-Functor Formalism for Solid Quasicoherent Sheaves II. Finish the proof of [6, Proposition 2.9.21] by handling the case of open immersions (see [8, Appendix to Lecture VIII]), thereby establishing the 6-functor formalism for solid quasicoherent sheaves on discrete adic spaces (and in particular schemes). Use it to prove Grothendieck duality on schemes (see [8, Theorem 11.1]) and that proper pushforward along noetherian schemes preserves coherent sheaves.

Talk 10: Analytic Adic Spaces I. Recall the definition of (complete) Huber pairs (see e.g. [11]). Following [1, §3.3] construct the analytic ring associated to a complete Huber pair and show its basic properties.

Talk 11: Analytic Adic Spaces II. Recall the definition of analytic adic spaces and its relation to rigid-analytic geometry (see e.g. [11]). Prove descent for solid quasicoherent sheaves on analytic adic spaces (see [1, Theorem 4.1]). If time permits, sketch consequences like descent of coherent sheaves and vector bundles, and finiteness of proper pushforward (see [3, §9, §12] for similar arguments in complex analytic geometry).

Talk 12: Hirzebruch-Riemann-Roch in Rigid Geometry. Formulate the Hirzebruch-Riemann-Roch theorem in rigid-analytic geometry and sketch its proof. The rigid-analytic proof is sketched in Clausen's notes in [7], while a much more detailed version of the proof (in the complex analytic setting) is presented in [3, §14, §15].

1. Condensed Sets

A *profinite set* is a topological space that can be written as a cofiltered limit of finite discrete sets (equipped with the limit topology). Recall the following properties:

Lemma 1.1. *(i) The category of profinite sets is equivalent to* Pro(Fin)*, i.e. the* Pro*-category of the category of finite sets. In other words, for profinite sets $S = \varprojlim_i S_i$ and $T = \varprojlim_j T_j$ we have*

$$\mathrm{Hom}(S,T) = \varprojlim_j \varinjlim_i \mathrm{Hom}(S_i, T_j).$$

(ii) A topological space is profinite if and only if it is compact Hausdorff and totally disconnected.
(iii) Every compact Hausdorff space admits a surjection from some profinite set.

Proof. For a quick discussion of Pro-categories see e.g. [5, §6.1], in particular [5, Remark 6.1.6]. The proof of the claimed Hom-identity in (i) reduces to the following observation: If $S = \varprojlim_i S_i$ is a profinite set and X is any discrete set then $\mathrm{Hom}(S, X) = \varinjlim_i \mathrm{Hom}(S_i, X)$, i.e. every continuous map $S \to X$ factors over some S_i (this observation will be important later!). Namely, the image of every map $f\colon S \to X$ is compact and hence finite and therefore corresponds to a disjoint decomposition of S into finitely many clopen subsets. One checks that this implies that f factors over some S_i (e.g. use a similar argument as in [10, Lemma 08ZZ]).

For (ii) see [8, Lemma 08ZY] and for (iii) see [8, Example 2.5] (extremally disconnected sets are in particular profinite). □

By Lemma 1.1.(i) profinite sets can be described without any reference to topology. As in [8, Definition 1.2] we now define condensed sets/abelian groups/rings as sheaves on the site of profinite sets. We will mostly ignore set-theoretic issues coming from the fact that the site of profinite sets is not small; see [8, Appendix to Lecture II] for a way to solve them. By Lemma 1.1.(iii) a condensed set is equivalently a sheaf on the site of compact Hausdorff spaces (cf. [8, Proposition 2.3]), i.e. it is determined by the "continuous" maps from compact Hausdorff spaces. We therefore get the following special case of condensed sets:

Definition 1.2. A topological space X is called *compactly generated* if it satisfies the following property: A subset $U \subseteq X$ is open if and only if for continuous map $f\colon S \to X$ from some compact Hausdorff space S, $f^{-1}(U) \subseteq S$ is open. (This is equivalent to the definition following [8, Example 1.5].)

The most prominent example of compactly generated topological spaces are first countable topological spaces X (as they are determined by maps $\mathbb{N} \cup \{\infty\} \to X$), see [8, Remark 1.6].

Proposition 1.3. *The assignment $X \mapsto [S \mapsto \mathrm{Hom}(S, X)]$ defines a faithful functor from the category of topological spaces to the category of condensed sets. It is fully faithful when restricted to the full subcategory of compactly generated topological spaces.*

Proof. This is an easy direct argument using the observation that if X is the condensed set associated to a topological space X' then $X(*)$ recovers the underlying set of X' (plus some diagram chases). See also the proof of [8, Proposition 1.7] for a slightly different and less direct way of formulating this (we will not need the adjoint functor $(-)_{\mathrm{top}}$ in this seminar). □

Recall the definition of injective and surjective maps of condensed sets (defined like on any site). On general sites surjectivity and images are hard to handle directly, but on the site of profinite sets one can make use of the existence of extremally disconnected sets (see [8, Definition 2.4] until [8, Proposition 2.7] and the discussion afterwards, explaining that the sheaf property on extremally disconnected sets is extremely simple). Deduce that a map $X \to Y$ is surjective if and only if $X(S) \to Y(S)$ is surjective for every extremally disconnected set S. Use this to compute $(\mathbb{R}/\mathbb{R}_{\mathrm{disc}}(S)$ for extremally disconnected sets S and verify that it is non-zero (see [8, Example 1.9]; note that for general profinite S we cannot show the claimed identity yet – this fill be part of the next talk).

If time permits, one can also introduce quasicompact and quasiseparated condensed sets and show that qcqs condensed sets are precisely the compact Hausdorff spaces (see the discussion following Definition 2.5 in [3]). In general, [3, §2] (until Definition 2.12) is a good alternative reference for some of the material of this talk.

2. Condensed Abelian Groups and Cohomology

Due to the existence of extremally disconnected sets, the category of condensed abelian groups has very nice formal properties (see [8, Theorem 2.2] including the proof). The compact projective generators are the free abelian groups $\mathbb{Z}[S]$ for extremally disconnected sets S, which will play an important role throughout the seminar.

Following the discussion after the proof of [8, Theorem 2.2] one can also introduce a tensor product $\otimes$ and an internal hom $\underline{\mathrm{Hom}}$ of condensed abelian groups with the usual adjunction – this can all be done on any site. Note that for condensed abelian groups M and N and any extremally disconnected set S we have $(M \otimes N)(S) = M(S) \otimes_{\mathbb{Z}(S)} N(S)$, where $\mathbb{Z}(S) = \mathcal{C}(S, \mathbb{Z})$ denotes the S-points of the discrete abelian group S (viewed as a condensed set); this description of the tensor product follows from the fact that the right-hand side sends finite disjoint unions of extremally disconnected sets to products and thus already satisfies the sheaf property on extremally disconnected sets.

As for sheaves on any site, there is a good notion of the derived category of condensed abelian groups and we also get derived versions of $\otimes$ and $\underline{\mathrm{Hom}}$.

After discussing the general properties of condensed abelian groups we study how they capture classical cohomology theories. Given a compact Hausdorff space S and a discrete abelian group M, recall the definitions of M-valued singular cohomology, sheaf cohomology and Čech cohomology of S and how they relate to each other (see the beginning of [8, §3]). Following [8, Theorem 3.2] and the discussion beforehand (excluding Proposition 3.1), introduce M-valued condensed cohomology of S as a fourth option and show that it is equivalent sheaf and Čech cohomology. In the proof of [8, Theorem 3.2], note that the key property of M is that $\Gamma(S, M) = \varinjlim_i \Gamma(S_i, M)$ for any profinite set $S = \varprojlim_i S_i$ (by discreteness of M). Use [8, Theorem 3.2] to compute $(\mathbb{R}/\mathbb{R}_{\mathrm{disc}})(S)$ for every profinite set S (see the middle of [9, p. 9], in particular footnote 5).

By [8, Theorem 3.2] it makes sense to define the condensed cohomology $H^i(X, M)$ (for every condensed set X and every condensed abelian groups M) as *the* M-valued cohomology of X; this generalizes the classical notion. As an example $R\Gamma(X, \mathbb{Z}_p) = \varprojlim_n R\Gamma(X, \mathbb{Z}/p^n)$. As a non-trivial example, sketch [8, Theorem 3.3] to compute the $\mathbb{R}$-valued cohomology of compact Hausdorff spaces.

3. Locally Compact Abelian Groups

Most topological abelian groups appearing in practice are locally compact, so this talk will focus on understanding them. The main focus lies on computing all derived internal homs between locally compact abelian groups. Follow [8, §4] up to the proof of [8, Theorem 4.3]. Here are some additional remarks on the material:

- In [8, Proposition 4.2] the crucial observation is that for compactly generated (weakly) Hausdorff topological spaces, the compact-open topology on the Hom-set provides an internal hom (i.e. a right adjoint to the product). This was one of the reasons for introducing compactly generated topological spaces in the first place.
- It may be helpful to reformulate [8, Corollary 4.8] in the following way: Given a map $M \to M'$ of condensed abelian groups and any condensed abelian group N then in order to show that the induced map $R\,\underline{\mathrm{Hom}}(M', N) \to R\,\underline{\mathrm{Hom}}(M, N)$ is an isomorphism, it is enough to show that for every extremally disconnected set S the map $R\Gamma(M' \times S, N) \to R\Gamma(M \times S, N)$ is an isomorphism. (This is a corollay of the spectral sequence given in [8, Corollary 4.8] and it is essentially the only way in which this spectral sequence is used – in fact, the resolution in [8, Theorem 4.5] is too inexplicit to efficiently use the spectral sequence for anything else.)
- To handle the case of infinite I in the proof of [8, Theorem 4.3.(i)], use the following general lemma (instead of [8, Proposition 3.1]):

  Lemma 3.1. *Let $K = \varprojlim_i K_i$ be a cofiltered limit of compact Hausdorff abelian groups. Then for every discrete abelian group M we have $R\,\underline{\mathrm{Hom}}(\varprojlim_i K_i, M) = \varinjlim_i R\,\underline{\mathrm{Hom}}(K_i, M)$.*

  Proof. By [8, Corollary 4.8] (or the above version of it) the statement reduces to the following: Given a cofiltered limit $X = \varprojlim_i X_i$ of compact Hausdorff spaces and any discrete abelian group M we have $R\Gamma(X, M) = \varinjlim_i R\Gamma(X_i, M)$. There are two ways to prove this: Firstly, by [8, Theorem 3.2] one can reduce this claim to a claim about classical sheaf cohomology of compact Hausdorff spaces (see the proof of [8, Proposition 3.1]. Secondly, there is a direct argument in the condensed world: If all X_i are profinite sets then all $H^k(X_i, M) = 0$ for $k > 0$ (by [8, Theorem 3.2]) and the claim reduces to the observation $M(X) = \varinjlim_i M(X_i)$ by discreteness of M; for general X_i one chooses functorial covers $S_i \twoheadrightarrow X_i$ by profinite sets S_i (e.g. via Stone-Čech compactifications) and computes $R\Gamma(X_i, M)$ via the associated Čech complexes (which works because all $S_i \times_{X_i} \cdots \times_{X_i} S_i$ are profinite and hence acyclic for M) – then the claim follows by using the fact that filtered colimits are exact. □

- In the proof of [8, Theorem 4.3.(i)], in the argument that the induced maps $H^i(\mathbb{R} \times S, M) \to H^i(S, M)$ are zero, one essentially uses the fact that $H^i(\mathbb{R} \times S, M)$ computes M-valued classical sheaf cohomology of $\mathbb{R} \times S$ and is therefore homotopy invariant. Unfortunately we only proved the comparison of cohomology theories for compact Hausdorff spaces and not $\mathbb{R} \times S$, but as shown in the proof of [8, Theorem 4.3.(i)] this result extends at least to $\mathbb{R} \times S$. One may contemplate how far one can generalize [8, Theorem 3.2].

Now, to compute all $R\,\underline{\mathrm{Hom}}$'s, we need a better understanding of compact Hausdorff abelian groups. In order to get this, we first note that the Pontrjagin duality functor even works in the derived sense if one plugs in a discrete abelian group:

Proposition 3.2. *Let M be a discrete abelian group. Then $\underline{\mathrm{Ext}}^i(M, \mathbb{R}/\mathbb{Z}) = 0$ for $i > 0$.*

Proof. The essential ingredient in the proof is the fact that for every extremally disconnected set S, the sheaf $\mathbb{R}/\mathbb{Z}$ on the topological space S is injective (this generalizes the fact that $\mathbb{R}/\mathbb{Z}$ is an injective object in the category of discrete abelian groups), see [4, Lemma VII.1.7] (we can probably skip the proof). Now fix an extremally disconnected set S. There is a natural morphism of sites

$$\alpha\colon \{\text{profinite sets over } S\} \to \{\text{clopen subsets of } S\}$$

and thus an associated pullback functor α^* from the category of abelian sheaves on S to the category of abelian sheaves on the site of profinite sets over S. This pullback functor is fully faithful on derived categories: This is a special case of the full faithfulness of the pullback from the étale to the pro-étale site on nice schemes (see [2, Remark 5.2.8]), but there is an easy direct argument in our case. Namely, the right adjoint of the pullback is the pushforward functor, which is t-exact in our case because (every clopen subset of) S is acyclic for condensed cohomology; but then it follows easily that the counit $\mathrm{id} \xrightarrow{\sim} \alpha_* \alpha^*$ is an isomorphism (even when viewed as functors on the derived categories) which implies the fully faithfulness of α^*.

With the full faithfulness of α^* at hand, we compute:

$$R\underline{\mathrm{Hom}}(M, \mathbb{R}/\mathbb{Z})(S) = R\mathrm{Hom}_{/S}(\alpha^* M, \mathbb{R}/\mathbb{Z}) = R\mathrm{Hom}_S(M, \alpha_*((\mathbb{R}/\mathbb{Z})|_S)).$$

Here $R\mathrm{Hom}_{/S}$ denotes the derived hom for abelian sheaves on the site of profinite sets over S and $R\mathrm{Hom}_S$ denotes the derived hom for abelian sheaves on S. The right-hand side is clearly zero in cohomological degrees above 0 because $\mathbb{R}/\mathbb{Z}$ is acyclic as a sheaf on S. □

As a special case of the proof of the proof of Proposition 3.2 we see that the derived category of discrete abelian groups embeds fully faithfully into the derived category of condensed abelian groups.

Corollary 3.3. *Let K be a compact Hausdorff abelian group. Then there is a short exact sequence of the form*

$$0 \to K \to \prod_I \mathbb{R}/\mathbb{Z} \to \prod_J \mathbb{R}/\mathbb{Z} \to 0$$

for some sets I and J (in the category of condensed abelian groups).

Proof. By Pontrjagin duality (see previous talk) we can write $K = \underline{\mathrm{Hom}}(M, \mathbb{R}/\mathbb{Z})$ for some discrete abelian group M. Pick a resolution $0 \to \bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z} \to M \to 0$ of M. Then we get a long exact sequence

$$0 \to \underline{\mathrm{Hom}}(M, \mathbb{R}/\mathbb{Z}) \to \underline{\mathrm{Hom}}(\bigoplus_I \mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to \underline{\mathrm{Hom}}(\bigoplus_J \mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to \underline{\mathrm{Ext}}^1(M, \mathbb{R}/\mathbb{Z}).$$

The last term is 0 by Proposition 3.2, so one easily arrives at the desired resolution of K. □

With the above results, one can decompose every locally compact abelian group into copies of $\mathbb{R}$, $\prod_I \mathbb{R}/\mathbb{Z}$ and discrete abelian groups. Thus by [8, Theorem 4.3] one can compute all $R\underline{\mathrm{Hom}}$'s between such groups, as illustrated by the following argument:

Proposition 3.4. *Let M and N be locally compact abelian groups. Then*

$$\underline{\mathrm{Ext}}^i(M, N) = 0, \qquad \text{for } i \geq 2.$$

Proof. This is a fun computation: By [8, Theorem 4.1.(i)] we can write $M = M' \times \mathbb{R}^m$ and $N = N' \times R^n$ for some $m, n \geq 0$ and some locally compact abelian groups M' and N' such that M' and N' have a compact open subgroup. It is thus enough to compute $R\underline{\mathrm{Hom}}(\mathbb{R}^m, \mathbb{R}^n)$, $R\underline{\mathrm{Hom}}(\mathbb{R}^m, N')$, $R\underline{\mathrm{Hom}}(M', \mathbb{R}^n)$ and $R\underline{\mathrm{Hom}}(M', N')$ and show that they all vanish above cohomological degree 1. Using [8, Theorem 4.3] we easily deduce $R\underline{\mathrm{Hom}}(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^{m \cdot n}$. By choosing a compact open subgroup $M_k \subseteq M'$ one gets a short exact sequence $0 \to M_k \to M' \to M_d \to 0$ such that M_d is discrete. It follows from Corollary 3.3 and [8, Theorem 4.3] that $R\underline{\mathrm{Hom}}(M_k, \mathbb{R}^n) = 0$ and hence $R\underline{\mathrm{Hom}}(M', \mathbb{R}^n) = R\underline{\mathrm{Hom}}(M_d, \mathbb{R}^n)$, which one can compute by choosing a 2-term resolution of M_d in terms of free abelian groups; in particular $R\underline{\mathrm{Hom}}(M', \mathbb{R}^n)$ vanishes above degree 1, as desired. We leave the remaining two cases to the reader. □

By a similar argument as above one can show that the $R\,\mathrm{Hom}$'s of locally compact abelian groups (in the sense of condensed mathematics) agree with previously ad-hoc defined $R\,\mathrm{Hom}$'s for locally compact abelian groups, see [8, Corollary 4.9]. This gives strong evidence that condensed mathematics provides the "right" model for topological abelian groups.

4. Solid Abelian Groups

In the previous talk we have seen that $R\,\underline{\mathrm{Hom}}$'s of condensed abelian groups are very nice. We now introduce a notion of "completeness" for condensed abelian groups in order to get better behavior for the tensor product. This talk follows [8, §5, §6], but we find the presentation in these sections a bit confusing, so in the following we provide an alternative route.

As in [8, Definition 5.1] define $\mathbb{Z}_\square[S] = \varprojlim_i \mathbb{Z}[S_i]$ (here we use slightly different notation from the reference in order to make it fit better with analytic rings later on) for profinite S and say that a condensed abelian group M is *solid* if for all S the natural map

$$\mathrm{Hom}(\mathbb{Z}_\square[S], M) \to \mathrm{Hom}(\mathbb{Z}[S], M) = M(S)$$

is an isomorphism. The goal of this talk is to prove that solid abelian groups form a nice category (cf. [8, Theorem 5.8]):

Theorem 4.1. *(i) The category of solid abelian groups is an abelian subcategory of all condensed abelian groups, stable under all limits, colimits and extensions. It is generated by the compact projective generators* $\mathbb{Z}_\square[S]$ *for profinite* S*. The inclusion* $\mathrm{Solid} \hookrightarrow \mathrm{Cond(Ab)}$ *has a left adjoint* $(-)^\square\colon \mathrm{Cond(Ab)} \to \mathrm{Solid}$ *mapping* $\mathbb{Z}[S] \mapsto \mathbb{Z}_\square[S]$*.*

(ii) The functor $\mathcal{D}(\mathrm{Solid}) \hookrightarrow \mathcal{D}(\mathrm{Cond(Ab)})$ *is fully faithful and identifies* $\mathcal{D}(\mathrm{Solid})$ *with the full subcategory of* $\mathcal{D}(\mathrm{Cond(Ab)})$ *consisting of those complexes all of whose cohomologies lie in* Solid*. The inclusion has a left adjoint* $(-)^{L\square}\colon \mathcal{D}(\mathrm{Cond(Ab)}) \to \mathcal{D}(\mathrm{Solid})$ *which is the left derived functor of* $(-)^\square$*.*

Remarks 4.2. (i) The category Solid of solid abelian groups inherits all the nice formal properties of Cond(Ab).

(ii) It will follow from the proof that a complex $M \in \mathcal{D}(\mathrm{Cond(Ab)})$ is solid (i.e. lies in $\mathcal{D}(\mathrm{Solid})$) if and only if for all profinite sets S the map $R\,\underline{\mathrm{Hom}}(\mathbb{Z}_\square[S], M) \to R\,\underline{\mathrm{Hom}}(\mathbb{Z}[S], M)$ is an isomorphism. This gives an alternative definition for solid abelian groups that works directly on the derived level.

To prove the theorem, we first make some formal reductions. In fact, the whole theorem follows from the following claim:

($*$) Given a map of the form $f\colon \bigoplus_{j\in J} \mathbb{Z}_\square[T_j] \to \bigoplus_{i \in I} \mathbb{Z}_\square[S_i]$ for profinite S_i, T_j and given any profinite set S the map $R\,\mathrm{Hom}(\mathbb{Z}_\square[S], \ker f) \xrightarrow{\sim} R\,\mathrm{Hom}(\mathbb{Z}[S], \ker f)$ is an isomorphism.

This is proved in detail in [8, Lemma 5.9], but the proof is not very enlightening so that we skip it. However, one observation is that it follows immediately from ($*$) that also $\operatorname{coker} f$ satisfies the same property as $\ker f$ (here it is important to work with $R\,\mathrm{Hom}$ instead of Hom) and the proof strategy is to show that then every solid abelian group can be written as $\operatorname{coker} f$ for suitable f (this eventually follows from the fact that every condensed abelian group can be written as a cokernel of a map $\bigoplus_j \mathbb{Z}[T_j] \to \bigoplus_i \mathbb{Z}[S_i]$). Note that ($*$) formally reduces to:

($*'$) Let $C = [\cdots \to C_n \to \cdots \to C_0 \to 0]$ be a complex in Cond(Ab) such that all C_n are direct sums of copies of $\mathbb{Z}[S]$ for various profinite sets S; then for every extremally disconnected set S the map $R\,\mathrm{Hom}(\mathbb{Z}_\square[S], C) \xrightarrow{\sim} R\,\mathrm{Hom}(\mathbb{Z}[S], C)$ is an isomorphism.

A detailed proof of this reduction is found in [8, Lemma 5.10], which we will again skip. The idea is to pick resolutions of $\ker f$.

It now remains to prove ($*'$). For this we need a better understanding of $\mathbb{Z}_\square[S]$. State [8, Theorem 5.4] (the proof can again be skipped – the general idea is to do a transfinite induction on the size of S using well-ordering). We deduce $\mathbb{Z}_\square[S] \cong \prod_I \mathbb{Z}$ for some set I, see [8, Corollary 5.5] (and the discussion following [8, Remark 5.3]). To prove ($*'$) we also need the following result:

Proposition 4.3. *Let K be a compact abelian group and $(M_i)_i$ a filtered system of condensed abelian groups. Then* $R\,\underline{\mathrm{Hom}}(K, \varinjlim_i M_i) = \varinjlim_i R\,\underline{\mathrm{Hom}}(K, M_i)$.

Proof. By [8, Corollary 4.8] (see previous talks) this reduces to the observation that $R\Gamma(K, \varinjlim_i M_i) = \varinjlim_i R\Gamma(K, M_i)$, which follows from the fact that K is qcqs as a condensed set. ☐

Proof of Theorem 4.1. The proof uses the solid understanding of $R\,\underline{\mathrm{Hom}}$'s of locally compact abelian groups, as discussed in the previous two talks. Let C and S be given as in $(*')$. First assume that C is concentrated in one degree, i.e. $C = \bigoplus \prod \mathbb{Z}$. Then we need to show that the map

$$R\,\underline{\mathrm{Hom}}(\mathbb{Z}_\square[S], \bigoplus \prod \mathbb{Z}) \to R\,\underline{\mathrm{Hom}}(\mathbb{Z}[S], \bigoplus \prod \mathbb{Z}) = \bigoplus \prod R\,\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z}).$$

is an isomorphism (here on the right-hand side we can pull out the direct sum because $\mathbb{Z}[S]$ is compact projective). The main challenge is that $\mathbb{Z}_\square[S]$ is not compact, so we cannot immediately pull out the direct sum on the left-hand side. To circumvent this issue, we write $\mathbb{Z}_\square[S] = \prod_I \mathbb{Z}$ for some set I and use the following triangle:

$$R\,\underline{\mathrm{Hom}}(\prod_I \mathbb{R}/\mathbb{Z}, \bigoplus \prod \mathbb{Z}) \to R\,\underline{\mathrm{Hom}}(\prod_I \mathbb{R}, \bigoplus \prod \mathbb{Z}) \to R\,\underline{\mathrm{Hom}}(\prod_I \mathbb{Z}, \bigoplus \prod \mathbb{Z}).$$

It remains to compute the two new terms in this triangle:

(a) We claim that $R\,\underline{\mathrm{Hom}}(\prod_I \mathbb{R}, \bigoplus \prod \mathbb{Z}) = 0$ (i.e. "there are not maps from $\mathbb{R}$-vector spaces to sums of products of $\mathbb{Z}$"). In fact, we can write $R\,\underline{\mathrm{Hom}}(\prod_I \mathbb{R}, \bigoplus \prod \mathbb{Z}) = R\,\underline{\mathrm{Hom}}_{\mathbb{R}}(\prod_I \mathbb{R}, R\,\underline{\mathrm{Hom}}(\mathbb{R}, \bigoplus \prod \mathbb{Z}))$, so it is enough to show that $R\,\underline{\mathrm{Hom}}(\mathbb{R}, \bigoplus \prod \mathbb{Z}) = 0$. Now use again the above triangle:

$$R\,\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, \bigoplus \prod \mathbb{Z}) \to R\,\underline{\mathrm{Hom}}(\mathbb{R}, \bigoplus \prod \mathbb{Z}) \to R\,\underline{\mathrm{Hom}}(\mathbb{Z}, \bigoplus \prod \mathbb{Z}) = \bigoplus \prod \mathbb{Z}.$$

By Proposition 4.3 and [8, Theorem 4.3] (discussed in previous talks) we have

$$R\,\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, \bigoplus \prod \mathbb{Z}) = \bigoplus \prod R\,\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, \mathbb{Z}) = \bigoplus \prod (\mathbb{Z}[-1]),$$

as desired.

(b) We can compute $R\,\underline{\mathrm{Hom}}(\prod_I \mathbb{R}/\mathbb{Z}, \bigoplus \prod \mathbb{Z})$ using the compactness of $\prod_I \mathbb{R}/\mathbb{Z}$: By Proposition 4.3 and [8, Theorem 4.3] we get

$$R\,\underline{\mathrm{Hom}}(\prod_I \mathbb{R}/\mathbb{Z}, \bigoplus \prod \mathbb{Z}) = \bigoplus \prod R\,\underline{\mathrm{Hom}}(\prod_I \mathbb{R}/\mathbb{Z}, \mathbb{Z}) = \bigoplus \prod (\bigoplus_I \mathbb{Z}[-1]).$$

Thus altogether we deduce

$$R\,\underline{\mathrm{Hom}}(\mathbb{Z}_\square[S], \bigoplus \prod \mathbb{Z}) = \bigoplus \prod (\bigoplus_I \mathbb{Z}).$$

It remains to observe that $R\,\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z}) = \bigoplus_I \mathbb{Z}$. Namely, $R\,\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z})(S') = R\Gamma(S \times S', \mathbb{Z}) = \mathcal{C}(S \times S', \mathbb{Z}) = \bigoplus_I \mathcal{C}(S', \mathbb{Z})$ using [8, Corollary 5.5].

We have prove $(*')$ in the case that C is concentrated in a single degree. The claim follows formally in the case that C is bounded. To deduce the general case we write C as the filtered colimit of its naive truncations (i.e. where we just replace C_n by 0 for $n \gg 0$). To pull this colimit out of the $R\,\underline{\mathrm{Hom}}$'s we need to show that $R\,\underline{\mathrm{Hom}}(\prod_I, C)$ is concentrated in degree $\le d$ for some d independent of C. This is proved by similar methods as above and uses all of the theory on locally compact abelian groups; we skip the details (see [8, p. 39, 40]). ☐

Remark 4.4. It follows from the proof that $\mathbb{R}^{L\square} = 0$ (cf. [8, Corollary 6.1.(iii)]). Namely, this amounts to saying that $R\,\mathrm{Hom}(\mathbb{R}, C) = 0$ for any C as in $(*')$, which is part of the computation. In particular, solid abelian groups do not capture any real analysis – the solid theory is fundamentally a non-archimedean theory.

Remark 4.5. In Solid, any object of the form $\prod_I \mathbb{Z}$ is compact projective because it is a retract of $\mathbb{Z}_\square[S]$ for some extremally disconnected set S. One can deduce from this fact that compact objects in Solid are stable under the solid tensor product (see below), which is an extremely useful property that is wrong in Cond(Ab) (and also wrong the liquid real analytic world).

Note that since Solid is stable under limits and colimits, it contains all discrete abelian groups (which are colimits of copies of $\mathbb{Z}$) and thus also groups like $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{Q}_p$-Banach spaces. We can finally introduce the promised tensor product:

Definition 4.6. There is a unique tensor product $\otimes_{\mathbb{Z}_\square}$ on Solid such that $(-)^\square\colon \mathrm{Cond(Ab)} \to \mathrm{Solid}$ commutes with tensor products. In particular $M \otimes_{\mathbb{Z}_\square} N = (M \otimes N)^\square$. (Here is crucial to have $R\,\underline{\mathrm{Hom}}(\mathbb{Z}_\square[S], M) = \dots$ instead of $R\,\mathrm{Hom}(\mathbb{Z}_\square[S], M) = \dots$.) A similar statement holds for derived categories, where the obtained tensor product is the derived tensor product $\otimes^L_{\mathbb{Z}_\square}$. See [8, Theorem 6.2] for details.

The main tool to compute solid tensor products is the following easy observation:

Proposition 4.7. *For any sets I and J we have $\prod_I \mathbb{Z} \otimes^L_{\mathbb{Z}_\square} \prod_J \mathbb{Z} = \prod_{I\times J} \mathbb{Z}$.*

Proof. By possibly enlarging I and J we can assume that $\prod_I \mathbb{Z} = \mathbb{Z}_\square[S]$ and $\prod_J \mathbb{Z} = \mathbb{Z}_\square[T]$ for some profinite sets S and T. Then

$$\mathbb{Z}_\square[S] \otimes^L_{\mathbb{Z}_\square} \mathbb{Z}_\square[T] = (\mathbb{Z}[S] \otimes \mathbb{Z}[T])^\square = \mathbb{Z}[S \times T]^\square = \mathbb{Z}_\square[S \times T] = \prod_{I \times J} \mathbb{Z},$$

as desired (cf. [8, Proposition 6.3]). □

As an easy consequence of Proposition 4.7 we deduce that $\mathbb{Z}[[x]] \otimes^L_{\mathbb{Z}_\square} \mathbb{Z}[[y]] = \mathbb{Z}[[x,y]]$, $\mathbb{Z}_p \otimes^L_{\mathbb{Z}_\square} \mathbb{Z}[[y]] = \mathbb{Z}_p[[y]]$ and

$$\mathbb{Z}_p \otimes^L_{\mathbb{Z}_\square} \mathbb{Z}_\ell = \begin{cases} \mathbb{Z}_p, & \text{if } p = \ell, \\ 0, & \text{else.} \end{cases}.$$

See [8, Example 6.4] for details.

If time permits, one can also discuss the Whitehead problem, a problem in abstract group theory from the 1950's: Given a discrete abelian group M such that $\mathrm{Ext}^1(M, \mathbb{Z}) = 0$, is M necessarily free? It was shown that the answer to this question depends on the model of set-theory. However, the following version is correct:

Proposition 4.8. *Let M be a discrete abelian group such that $\underline{\mathrm{Ext}}^1(M, \mathbb{Z}) = 0$. Then M is free.*

Proof. Pick a resolution $0 \to \bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z} \to M \to 0$. Applying $R\,\underline{\mathrm{Hom}}(-, \mathbb{Z})$ we obtain a short exact sequence $0 \to \underline{\mathrm{Hom}}(M, \mathbb{Z}) \to \prod_I \mathbb{Z} \to \prod_J \mathbb{Z} \to 0$ (here we use that $\underline{\mathrm{Ext}}^1(M, \mathbb{Z}) = 0$). This is a short exact sequence of solid abelian groups and $\prod_J \mathbb{Z}$ is projective in Solid, hence this sequence splits. Now apply $\underline{\mathrm{Hom}}(-, \mathbb{Z})$ again: By using the fact that $R\,\underline{\mathrm{Hom}}(\prod \mathbb{Z}, \mathbb{Z}) = \bigoplus \mathbb{Z}$ we see that we get the original resolution of M back. Therefore this resolution splits and hence M is free. □

5. ∞-Categories for the Working Mathematician

This talk will be prepared by the organizers of the seminar.

6. Analytic Rings

In this case we generalize solid abelian groups to the relative setting: We consider condensed rings equipped with some notion of "complete" modules. As in the case of solid modules, this "completeness" notion will be defined by specifying new free objects (like replacing $\mathbb{Z}[S]$ by $\mathbb{Z}_\square[S]$). Analytic rings are introduced in [8, §7, §8] (the non-derived version; be careful with this, see Remark 6.3) and [9, §12]. These references may be a bit hard to digest, so we present a summary of the main definitions and results in the following. The definition of analytic rings is as follows (see [9, Definitions 12.1, 12.9, 12.10]):

Definition 6.1. An *analytic ring* is a pair $\mathcal{A} = (\underline{\mathcal{A}}, \mathcal{M}_\mathcal{A})$, where $\underline{\mathcal{A}}$ is an (animated) condensed ring and $\mathcal{M}_\mathcal{A}$ is a functor

$$\mathcal{M}_\mathcal{A}\colon \{\text{extr. disc. sets}\} \to \mathcal{D}^{\leq 0}(\underline{\mathcal{A}}), \qquad S \mapsto \mathcal{M}_\mathcal{A}(S) =: \mathcal{A}[S]$$

together with a natural transformation $\underline{\mathcal{A}}[S] \to \mathcal{A}[S]$ such that the following properties are satisfied:

(i) $\mathcal{M}_{\mathcal{A}}$ transforms finite disjoint unions into products and the map $\underline{\mathcal{A}} \to \mathcal{A}[*] = \mathcal{M}_{\mathcal{A}}(*)$ is an isomorphism of $\underline{\mathcal{A}}$-modules.

(ii) If $M \in \mathcal{D}(\underline{\mathcal{A}})$ is a sifted colimit of copies of $\mathcal{A}[S]$ for varying extremally disconnected sets S then for every extremally disconnected set S the natural map

$$\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], M) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], M)$$

is an isomorphism.

(iii) For every prime p, the forgetful functor along the Frobenius $\underline{\mathcal{A}} \to \underline{\mathcal{A}}/p$ maps every $(\mathcal{A}/p)[S] := \mathcal{A}[S]/p$ to an $\underline{\mathcal{A}}$-module M satisfying the property in (ii).

Remark 6.2. Let us explain conditions (ii) and (iii) in Definition 6.1: In (ii), recall that sifted colimits are a combination of filtered colimits and geometric realizations. Hence the module M appearing in (ii) is an analog of the complex C appearing in condition $(*')$ of talk 5, i.e. a right-bounded complex all of whose terms are direct sums of copies of $\mathcal{A}[S]$ for varying S. Since $\mathcal{A}[S]$ can itself be a complex in general, we cannot directly work with complexes C, so instead we use the sifted colimits. Apart from this detail, condition (ii) captures the same thing as condition $(*')$ from talk 5.

Condition (iii) is more mysterious and may be an artifact of the current way the theory is built. It is required for technical reasons to solve the following problem: In practice one sometimes constructs "uncomplete analytic rings", which are the same as analytic rings but where possibly the map $\underline{\mathcal{A}} \to \mathcal{A}[*]$ is not an isomorphism. One then needs a way of "completing" such an uncomplete analytic ring, which roughly just replaces $\underline{\mathcal{A}}$ by $\mathcal{A}[*]$. However, to make this work we need to define the structure of an animated ring on $\mathcal{A}[*]$ (which a priori is only an $\underline{\mathcal{A}}$-module) and this is surprisingly subtle; as it turns out, the condition on the Frobenius is enough to make this construction work (see the proof of [9, Proposition 12.26]). In practice one does not need to worry about (iii) too much: It is preserved under all constructions of analytic rings and it is automatic if for some integer m all $\mathcal{A}[S]$ are m-truncated for all profinite sets S (here it is not enough to only require extremally disconnected sets S) or if all $\mathcal{A}[S]$ are solid as a (derived) abelian group, see [9, Proposition 12.24].

Remark 6.3. Using the explanations from the previous remark, we note that as a special case of analytic rings we can define static (i.e. "underived") analytic rings as follows: It is a pair $\mathcal{A} = (\underline{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$ where $\underline{\mathcal{A}}$ is a static condensed ring and $\mathcal{M}_{\mathcal{A}} \colon S \to \mathcal{A}[S]$ is a functor from profinite sets to static $\underline{\mathcal{A}}$-modules satisfying condition (i) above and additionally the following variant of (ii): Given any complex $C = [\cdots \to C_n \to \cdots \to C_0 \to 0]$ of static $\underline{\mathcal{A}}$-modules of the form $C_n = \bigoplus_i \mathcal{A}[S_i]$ for varying profinite sets S_i, and given any profinite set S, the natural map

$$\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C)$$

is an isomorphism (where $\underline{\mathrm{Hom}}$ denotes the derived $\underline{\mathrm{Hom}}$ by convention!). This is close to the definition in [8, Definitions 7.1, 7.4], although we use profinite sets instead of extremally disconnected sets in order to make $\mathcal{A}$ truly static (in the sense that $\mathcal{A}[S]$ is static for every profinite set S), thus making condition (iii) of Definition 6.1 automatic.

Remark 6.4. In practice most of the analytic rings one encounters are static, so it is usually sufficient to work with the definition in Remark 6.3. However, some general constructions (like tensor products of analytic rings) may result in non-static analytic rings, and it is important to keep track of those as well.

With the notion of analytic rings at hand, let us introduce the "complete" modules over an analytic ring (see [9, Definition 12.3] and [8, Proposition 7.5]):

Definition 6.5. Let $\mathcal{A}$ be an analytic ring. An *$\mathcal{A}$-module* is an $\underline{\mathcal{A}}$-module $M \in \mathcal{D}(\underline{\mathcal{A}})$ such that for every extremally disconnected set S the map

$$\underline{\mathrm{Hom}}(\mathcal{A}[S], M) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\underline{\mathcal{A}}[S], M)$$

is an isomorphism. We denote $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\underline{\mathcal{A}})$ the full subcategory spanned by the $\mathcal{A}$-modules.

As in the proof of Theorem 4.1 one can formally deduce from condition (ii) of Definition 6.1 that $\mathcal{A}$-modules satisfy the following nice properties (see also [9, Proposition 12.4] and [8, Proposition 7.5]):

Theorem 6.6. *Let $\mathcal{A}$ be an analytic ring. The ∞-category $\mathcal{D}(\mathcal{A})$ is stable under all limits and colimits in $\mathcal{D}(\underline{\mathcal{A}})$ and generated under colimits by the compact objects $\mathcal{A}[S]$ for extremally disconnected sets S. The t-structure on $\mathcal{D}(\underline{\mathcal{A}})$ restricts to a t-structure on $\mathcal{D}(\mathcal{A})$ (i.e. containment in $\mathcal{D}(\mathcal{A})$ can be checked on cohomology). The inclusion $\mathcal{D}(\mathcal{A}) \hookrightarrow \mathcal{D}(\underline{\mathcal{A}})$ admits a left adjoint*

$$- \otimes_{\underline{\mathcal{A}}} \mathcal{A} \colon \mathcal{D}(\underline{\mathcal{A}}) \to \mathcal{D}(\mathcal{A})$$

mapping $\underline{\mathcal{A}}[S] \mapsto \mathcal{A}[S]$.

Proof. As mentioned above, this follows formally from condition (ii) of Definition 6.1. The details can be skipped, but we refer the interested reader to [6, Proposition 2.3.2.(i)] for a rigorous proof (albeit in the more general “almost” setting). □

Definition 6.7. Let $\mathcal{A}$ be an analytic ring. There is a unique tensor product $\otimes_{\mathcal{A}}$ on $\mathcal{D}(\mathcal{A})$ such that $- \otimes_{\underline{\mathcal{A}}} \mathcal{A}$ commutes with tensor products.

Remark 6.8. If $\mathcal{A}$ is a static analytic ring (as defined in Remark 6.3) then there is also a version of Theorem 6.6 on the abelian level, see [8, Proposition 7.5.(i)].

It is high time for some basic examples of analytic rings (cf. [8, Examples 7.3]):

Examples 6.9.
(a) Let A be any (animated) condensed ring. Then there is an associated analytic ring $A = (A, S \mapsto A[S])$ whose “complete” modules are just all A-modules.
(b) $\mathbb{Z}_\square = (\mathbb{Z}, S \mapsto \mathbb{Z}_\square[S])$ is an analytic ring, as proved in talk 5. We have $\mathcal{D}(\mathbb{Z}_\square) = \mathcal{D}(\text{Solid})$, the ∞-category of (derived) solid abelian groups.
(c) Fix a real number p with $0 < p \leq 1$. One can define an analytic ring structure $\mathbb{R}_{\sim p} = (\mathbb{R}, S \mapsto \mathcal{M}_{<p}(S))$, where $\mathcal{M}_{<p}(S)$ is a space of certain “$\mathbb{R}$-valued measures” (see the discussion starting in the middle of [3, p. 27]). It is a deep result by Clausen-Scholze that this is indeed an analytic ring (i.e. satisfies condition (ii) above, cf. [9, Theorem 6.5]). The associated “complete” condensed $\mathbb{R}$-vector spaces are called the *p-liquid* $\mathbb{R}$-vector spaces. The induced “complete” tensor product is called the *p-liquid tensor product.*

There are some general constructions how to produce new analytic rings out of existing ones. In order to state these, we need to make analytic rings into an ∞-category:

Definition 6.10.
(a) A morphism of analytic rings $\mathcal{A} \to \mathcal{B}$ is a morphism $\underline{\mathcal{A}} \to \underline{\mathcal{B}}$ of underlying (animated) condensed rings such that the induced forgetful functor $\mathcal{D}(\underline{\mathcal{B}}) \to \mathcal{D}(\underline{\mathcal{A}})$ maps $\mathcal{D}(\mathcal{B})$ to $\mathcal{D}(\mathcal{A})$. We denote the ∞-category of analytic rings by AnRing.
(b) Given a morphism $\mathcal{A} \to \mathcal{B}$, the forgetful functor $\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ has a left adjoint

$$- \otimes_{\mathcal{A}} \mathcal{B} \colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$$

which commutes with the tensor products. It can be computed as $M \otimes_{\mathcal{A}} \mathcal{B} = (M \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{B}}) \otimes_{\underline{\mathcal{B}}} \mathcal{B}$.

We now come to the promised general constructions of analytic rings. The first two constructions are “cheap” ways of getting additional structure on an analytic ring and the other constructions use colimits in AnRing.

Proposition 6.11.
(i) Let $\mathcal{A}$ be an analytic ring and let B be any condensed $\underline{\mathcal{A}}$-algebra whose underlying $\underline{\mathcal{A}}$-module is an $\mathcal{A}$-module. Then the pair

$$B_{\mathcal{A}/} := (B, S \mapsto B \otimes_{\mathcal{A}} \mathcal{A}[S])$$

defines an analytic ring. We call this the induced *analytic structure on B. A B-module $M \in \mathcal{D}(B)$ lies in $\mathcal{D}(B_{\mathcal{A}/})$ if and only if the underlying $\underline{\mathcal{A}}$-module lies in $\mathcal{D}(\mathcal{A})$.*
(ii) Let A be an (animated) condensed ring. There is a bijective correspondence between analytic ring structures $\mathcal{A}$ on A and analytic ring structures $\mathcal{A}_0$ on $\pi_0 A$. Under this correspondence, an A-module M is an $\mathcal{A}$-module if and only if all $H^i(M)$ are $\mathcal{A}_0$-modules.

Proof. To prove (i) we use the fact that $\underline{\mathrm{Hom}}_B(B \otimes_{\mathcal{A}} \mathcal{A}[S], M) = \underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], M)$ to easily deduce condition (ii) of Definition 6.1. (We ignore condition (iii), which can also be easily checked.)

In (ii), the second part of the claim already tells us what the associated ∞-categories of modules need to look like, which proves uniqueness of the analytic ring structures. For the existence one needs to explicitly construct $\mathcal{A}[S]$, for which we refer to [9, Proposition 12.21]. This proof can be skipped as we do not care much about the derived structures. $\square$

Example 6.12. Let A be any discrete ring. Then we can define $(A, \mathbb{Z})_\square = (A, S \mapsto A \otimes \mathbb{Z}_\square[S])$, which defines an analytic ring (this also appears as [8, Proposition 7.9.(ii)]). An $(A, \mathbb{Z})_\square$-module is an A-module whose underlying abelian group is solid. Note that if A is static then $(A, \mathbb{Z})_\square[S]$ is static for every profinite set S (see the beginning of [8, Appendix to Lecture VIII]).

Example 6.13. For every prime p the pair $\mathbb{Z}_{p,\square} := (\mathbb{Z}_p, S \mapsto \varprojlim_i \mathbb{Z}_p[S_i])$ is an analytic ring (see [8, Proposition 7.9.(i)]). Namely, we compute

$$\mathbb{Z}_{p,\square}[S] = \varprojlim_i \mathbb{Z}_p[S_i] = \varprojlim_i \underline{\mathrm{Hom}}(\mathcal{C}(S_i, \mathbb{Z}), \mathbb{Z}_p) = \underline{\mathrm{Hom}}(\varinjlim_i \mathcal{C}(S_i, \mathbb{Z}), \mathbb{Z}_p) = \underline{\mathrm{Hom}}(\mathcal{C}(S, \mathbb{Z}), \mathbb{Z}_p) \cong \prod_I \mathbb{Z}_p,$$

for some set I depending on S. By Proposition 4.7 we deduce

$$\mathbb{Z}_{p,\square}[S] = \prod_I \mathbb{Z}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}_\square} \prod_I \mathbb{Z} = \mathbb{Z}_p \otimes_{\mathbb{Z}_\square} \mathbb{Z}_\square[S],$$

hence $\mathbb{Z}_{p,\square}$ is just the induced analytic ring structure on $\mathbb{Z}_p$.

One can also construct new analytic rings from existing ones using colimits. In fact, the ∞-category AnRing has all small colimits, and these colimits have a rather explicit description: see [9, Proposition 12.12] and [6, Proposition 2.3.15] (with $V = \mathbb{Z}$). While sifted (e.g. filtered) colimits are easy to describe, pushouts (i.e. tensor products) are a bit subtle to define.

In the next talk we will generalize the example $(A, \mathbb{Z})_\square$ to $(B, A)_\square$ for a general map $A \to B$ of discrete rings. The crucial input is that if A is a static discrete ring that is of finite type over $\mathbb{Z}$ then $A_\square = (A, S \mapsto \varprojlim_i A[S_i])$ is an analytic ring.

7. Discrete Adic Spaces

The goal of this talk is to define and study the analytic rings $(B, A)_\square$ for every map of discrete rings $A \to B$. By "gluing" these rings we obtain the theory of discrete adic spaces, which form a generalization of schemes. While one is usually mostly interested in schemes, having discrete adic spaces can sometimes be very useful to simplify some constructions, which we will see next time.

We will define $(B, A)_\square$ as the induced analytic ring structure on B coming from $A_\square$, so we first need to define $A_\square$. This can be done as follows (see also [8, Theorem 8.1]):

Theorem 7.1. *Let A be a static discrete ring which is of finite type over $\mathbb{Z}$. Then the pair $A_\square := (A, S \mapsto \varprojlim_i A[S_i])$ is an analytic ring.*

Proof. Note that $A_\square[S] \cong \prod_I A$ by the same argument as for $\mathbb{Z}$. Now if $A \to B$ is a finite map of finite-type $\mathbb{Z}$-algebras then $B_\square = (B, A)_\square$ has the induced analytic structure from $A_\square$, i.e. $B_\square[S] = B \otimes_A A_\square[S]$ for all profinite sets S (see the beginning of [8, Appendix to Lecture VIII]). Thus if $A_\square$ is an analytic ring then also $B_\square$ is an analytic ring (see previous talk). We can therefore reduce to the case $A = \mathbb{Z}[x_1, \ldots, x_n]$. By induction, we reduce to the following claim:

(∗∗) Suppose that A is a finite-type $\mathbb{Z}$-algebra such that $A_\square$ is an analytic ring; then $A[x]_\square$ is an analytic ring.

Now we argue similarly as in the case of $\mathbb{Z}_\square$: Pick a complex $C = [\cdots \to C_n \to \cdots \to C_0 \to 0]$ of static $A[x]$-modules such that each C_n is of the form $C_n = \bigoplus \prod A[x]$ and fix some profinite set S. We need to show that the map

$$\underline{\mathrm{Hom}}_{A[x]}(A[x]_\square[S], C) \to \underline{\mathrm{Hom}}_{A[x]}(A[x][S], C) = \underline{\mathrm{Hom}}_{A[x]}(A[x] \otimes_A A_\square[S], C)$$

is an isomorphism (here the second equality comes from the fact that C is solid over A). Consider the $A[x]$-algebra $A[x]_\infty := A((x^{-1}))$, which will play the role of $\mathbb{R}$. In fact, one checks that the cokernel Q of the injective map $A[x] \otimes_A A_\square[S] \to A[x]_\square[S]$ is an $A[x]_\infty$-module (see [8, Observation 8.9]), so that

$$\underline{\mathrm{Hom}}_{A[x]}(Q, C) = \underline{\mathrm{Hom}}_{A[x]_\infty}(Q, \underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, C)).$$

Our goal is to show that this is zero, which reduces to showing that $\underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, C) = 0$. But $A[x]_\infty$ is compact in $\mathcal{D}((A[x], A)_\square)$ (see [8, Observation 8.6]), so we can pull out colimits in C to reduce to the case $C = A[x]$, which is an easy explicit computation (see [8, Observation 8.8]). ☐

Definition 7.2. (a) Let A be a discrete (animated) ring. Then we define $A_\square := \varinjlim_i A_{i,\square}$, where $A = \varinjlim_i A_i$ is any representation of the animated ring A as a sifted colimit of static discrete rings that are of finite type over $\mathbb{Z}$ (if A is static then we can even choose filtered such colimit). In other words, we have $A_\square[S] = \varinjlim_i \prod_J A_i$ for a suitable set J.

(b) Let $A \to B$ be a map of discrete (animated) rings. Then we define $(B, A)_\square := B_{A_\square/}$ as the induced analytic ring, i.e. $(B, A)_\square[S] = B \otimes_A A_\square[S]$.

(c) We denote $D_\square(A) := \mathcal{D}(A_\square)$ and $\mathcal{D}_\square(B, A) := \mathcal{D}((B, A)_\square)$. Recall that $\mathcal{D}_\square(B, A)$ is the full subcategory of $\mathcal{D}(B)$ consisting of those condensed B-modules which are solid over A.

In practice one is mostly interested in the analytic rings $A_\square$, but it is useful to allow the more general version $(B, A)_\square$. A natural question to ask is how much $(B, A)_\square$ depends on A. We get:

Lemma 7.3. *Let $A \to B$ be a map of static discrete (i.e. classical) rings and let $\tilde{A} \subseteq B$ denote the integral closure of the image of A. Then $(B, A)_\square = (B, \tilde{A})_\square$.*

Proof. See [6, Lemma 2.9.4]. ☐

This leads us to the definition of discrete Huber pairs (A, A^+) as in [6, Definition 2.9.3.(a)-(b)]. In fact, the association $(A, A^+) \mapsto (A, A^+)_\square$ defines a fully faithful functor from the category of discrete Huber pairs to AnRing (see [6, Proposition 2.9.6.(i)] – we will postpone the proof of this to a later talk (where we show this also for non-discrete Huber pairs).

We now study the analytic rings $(A, A^+)_\square$. We first show that the solid base-change can be computed quite explicitly: If we have an integral map $A \to B$ of classical rings then $- \otimes_{A_\square} B_\square = - \otimes_A B$ (because $B_\square = (B, A)_\square$). Thus in order to understand base-change along solid analytic rings in general, it remains to handle the case of polynomial rings:

Lemma 7.4. *Let A be a classical ring which is of finite type over $\mathbb{Z}$ and denote $A[x]_\infty := A((x^{-1}))$. Then there is a natural equivalence of functors*

$$- \otimes_{A_\square} A[x]_\square = \underline{\mathrm{Hom}}_A(A[x]_\infty / A[x], -).$$

In particular, $- \otimes_{A_\square} A[x]_\square$ is t-exact and preserves all small limits.

Proof. Using adjunctions one can construct a morphism from left to right. Both sides commute with colimits, so the desired equivalence can be checked on the generators $\prod A$. The left-hand side will send this to $\prod A[x]$. The right-hand side sends it to $\prod \underline{\mathrm{Hom}}_A(A[x]_\infty/A[x], A)$, so it remains to show that $\underline{\mathrm{Hom}}_A(A[x]_\infty/A[x], A) = A[x]$. But $A[x]_\infty/A[x] = x^{-1}A[[x^{-1}]] \cong \prod_{\mathbb{N}} A$ is itself isomorphic to a solid generator, from which one easily deduces the claim (see [1, Proposition 3.13] for details). ☐

By combining Lemmas 7.3 and 7.4 one can prove many basic properties of the base-change along maps $(A, A^+)_\square \to (B, B^+)_\square$: By [6, Proposition 2.9.6.(ii)] the inclusion of discrete Huber pairs into analytic rings is compatible with non-empty colimits (in particular tensor products) and by [6, Proposition 2.9.7.(ii)] every map $(A, A^+)_\square \to (B, B^+)_\square$ is *steady*, i.e. satisfies the usual pullback relation (as in [6, Corollary 2.9.8]).

We now want to "glue" the analytic rings $(A, A^+)_\square$ into discrete adic spaces. This gluing procedure is completely formal, once we specify a class of "standard open immersions" $(A, A^+)_\square) \to (B, B^+)_\square$. This is explained in detail in [6, §2.4] (in much more generality than what we need), see in particular [6, Definitions 2.4.8–2.4.10]. The construction of discrete adic spaces and schemes is then carried out in [6, Definitions 2.9.13 and 2.9.14]. It is easy to relate these constructions to more classical definitions (via the Yoneda embedding), but one subtlety arises: In the analytic theory, we define open coverings via jointly conservative pullbacks, whereas in classical constructions one uses underlying topological spaces to define the coverings. One thus needs to check that both notions of covering agree, which is the content of [6, Lemma 2.9.19].

Remark 7.5. We define discrete adic spaces in a slightly different way than Huber's original definition. Namely, we do *not* deem the map $f\colon \operatorname{Spa}(\mathbb{Z}[T,T^{-1}],\mathbb{Z}[T]) \hookrightarrow \operatorname{Spa}(\mathbb{Z}[T],\mathbb{Z}[T])$ an open immersion (instead this becomes a closed immersion). It turns out that with this definition the theory of discrete adic spaces becomes more elegant (and has a closer connection to schemes: any open subspace of $\operatorname{Spa}(A,A)$ is of the form $\operatorname{Spa}(B,B)$, i.e. the open subspaces of $\operatorname{Spa}(A,A)$ and $\operatorname{Spec} A$ agree). Most importantly though, the 6-functor formalism for solid quasicoherent sheaves dictates that f cannot be an open immersion. This new definition of (discrete) adic spaces first appeared in [3, Theorem 7.4].

8. The 6-Functor Formalism for Solid Quasicoherent Sheaves I

Starting from [6, Lemma 2.9.22], introduce canonical compactifications of discrete adic spaces as well as +-finite type, separated and proper maps and show how they all relate to each other. Also introduce solid quasicoherent sheaves on discrete adic spaces via gluing (see the paragraph right before [6, Proposition 2.9.31] and [8, Proposition 10.5]). Then explain the general notion of 6-functor formalisms (see also [6, §A.5], but the precise ∞-categorical definition is not that important) and formulate its incarnation for solid quasicoherent sheaves on discrete adic spaces (see [6, Proposition 2.9.31]). Sketch how this result reduces to showing that the pullbacks along $\operatorname{Spec}\mathbb{Z}[x] \hookrightarrow \operatorname{Spa}(\mathbb{Z}[x],\mathbb{Z})$ and $\operatorname{Spec}\mathbb{Z}[x,x^{-1}] \hookrightarrow \operatorname{Spec}\mathbb{Z}[x]$ have a left adjoint satisfying the projection formula (feel free to ignore the set-theoretic issues involving the cardinals κ).

9. The 6-Functor Formalism for Solid Quasicoherent Sheaves II

Show that the pullback functors along the basic open immersions $\operatorname{Spec}\mathbb{Z}[x] \hookrightarrow \operatorname{Spa}(\mathbb{Z}[x],\mathbb{Z})$ and $\operatorname{Spec}\mathbb{Z}[x,x^{-1}] \hookrightarrow \operatorname{Spec}\mathbb{Z}[x]$ have a left adjoint satisfying the projection formula, thereby finishing the proof of [6, Proposition 2.9.31]. A reference for the former case is the discussion following (and including) [8, Observation 8.10]. This is already enough to construct the 6-functor formalism, but it is a priori not clear that it has the expected form for all open immersions (i.e. $j^! = j^*$ if j is an open immersion). These require two more computations:

Lemma 9.1. *Let $A \to B$ be a map of discrete rings with associated map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes. Assume that one of the following is true:*

(i) $B = A[x]$.
(ii) B is perfect as an A-module.

Then the natural transformation $f^! A \otimes f^* \xrightarrow{\sim} f^!$ of functors $\mathcal{D}_\square(A) \to \mathcal{D}_\square(B)$ is an isomorphism. Moreover, in case (i) we have $f^! A = B[1]$ and in case (ii) we have $f^! A = \underline{\operatorname{Hom}}_A(B,A)$, so in both cases $f^! A$ is discrete and perfect as a B-module.

Proof. We first prove (i), so assume $B = A[x]$. By playing around with adjunctions we see that for every $M \in \mathcal{D}_\square(A)$ we have

$$f^! M = \underline{\operatorname{Hom}}_{A[x]}(A[x], M) = \underline{\operatorname{Hom}}_A(f_!(A[x]), M)$$

as A-modules. Denoting $f_0\colon \operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}$ the projection, we get by proper base-change that $f_!(A[x]) = f_{0!}(\mathbb{Z}[x]) \otimes_{\mathbb{Z}_\square} A_\square$. To compute $f_{0!}(\mathbb{Z}[x])$ we factor this map as $\operatorname{Spec}\mathbb{Z}[x] \hookrightarrow \operatorname{Spa}(\mathbb{Z}[x],\mathbb{Z}) \to \operatorname{Spec}\mathbb{Z}$, where the first map is an open immersion and the second map is proper. We know how to compute the lower shriek along the first map (see [8, Observation 8.11], which was used above) and along the second map it is just the forgetful functor, so altogether we obtain $f_{0!}(\mathbb{Z}[x]) = \mathbb{Z}[x]_\infty/\mathbb{Z}[x] \cong \prod_{\mathbb{N}} \mathbb{Z}$, where $\mathbb{Z}[x]_\infty = \mathbb{Z}((x^{-1}))$. Using Lemma 7.4 we compute

$$\begin{aligned} f^! M &= \underline{\operatorname{Hom}}_A(f_!(A[x]), M) = \underline{\operatorname{Hom}}_{\mathbb{Z}}(f_{0!}(\mathbb{Z}[x]), M) = \underline{\operatorname{Hom}}_{\mathbb{Z}}(\mathbb{Z}[x]_\infty/\mathbb{Z}[x], M)[1] = (M \otimes_{\mathbb{Z}_\square} \mathbb{Z}[x]_\square)[1] \\ &= f_0^* M[1] = f^* M[1]. \end{aligned}$$

This proves (i).

Part (ii) is very simple: If B is perfect as an A-module then the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is proper (recall that this can be checked on the underlying classical schemes and the map $\pi_0 A \to \pi_0 B$ is finite) and thus $f_! = f_*$. It follows that for every $M \in \mathcal{D}_\square(A)$ we have

$$f^! M = \underline{\operatorname{Hom}}_B(B, f^! M) = \underline{\operatorname{Hom}}_A(f_!, M) = \underline{\operatorname{Hom}}_A(B, M) = \underline{\operatorname{Hom}}_A(B, A) \otimes_A M,$$

where in the last equality we used that B is perfect. □

Corollary 9.2. *If $j\colon U \hookrightarrow X$ is an open immersion of discrete adic spaces then $j^! = j^*$ as functors $\mathcal{D}_\square(X) \to \mathcal{D}_\square(U)$.*

Proof. This reduces to the case $X = \operatorname{Spec}\mathbb{Z}[x]$ and $U = \operatorname{Spec}\mathbb{Z}[x, x^{-1}]$. Then factor j as $\operatorname{Spec}\mathbb{Z}[x, x^{-1}] \hookrightarrow \operatorname{Spec}\mathbb{Z}[x, y] \to \operatorname{Spec}\mathbb{Z}[x]$ and apply Lemma 9.1 to both maps to easily arrive at the desired result. □

Note that in all of the above, the only actual computation we needed to make is for the polynomial ring over $\mathbb{Z}$ – everything else follows by abstract nonsense. We can now generalize Lemma 9.1 to a more geometric setting:

Definition 9.3. A map $f\colon Y \to X$ of schemes is called *perfect* if on suitable affine open covers of X and Y it can be constructed using compositions of maps of the two types in Lemma 9.1.

Remark 9.4. For classical (i.e. non-derived) schemes the notion of perfect morphisms is for example studied in [10, §0685]. The derived version of perfect morphisms has the advantage that it is stable under any base-change. We do not know if the derived version of perfect morphisms has been studied in the literature and we do not want to pursue that study here (for example, the above definition is not very nice; there should be a more axiomatic one). The most important case of perfect morphisms are finite-type finite-Tor dimension maps of noetherian schemes (and any base-change of such); in particular smooth maps are perfect.

Proposition 9.5. *Let $f\colon Y \to X$ be a perfect map of schemes. Then:*

(i) The natural transformation $f^!\mathcal{O}_X \otimes f^* \xrightarrow{\sim} f^!$ of functors $\mathcal{D}_\square(X) \to \mathcal{D}_\square(Y)$ is an isomorphism and $f^!\mathcal{O}_X$ is discrete and perfect.

(ii) Given any map $g\colon X' \to X$ with base-change

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & X' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

the natural transformation $g'^* f^! \xrightarrow{\sim} f'^! g^*$ of functors $\mathcal{D}_\square(X) \to \mathcal{D}_\square(Y')$ is an isomorphism.

Proof. Both statements can be checked locally on X and Y (using Corollary 9.2) and are stable under compositions in f, so we can assume that X and Y are affine and f is of one of the two types in Lemma 9.1. Then (i) follows immediately from Lemma 9.1 and (ii) is an easy computation on both cases, using similar methods as in the proof of Lemma 9.1. □

See also [8, Proposition 11.4] for a slightly different version of Proposition 9.5. Following [8, Proposition 11.5 and Theorem 11.6], we see that for smooth f of pure dimension d we have $f^!\mathcal{O}_X = \omega_f[d]$ (feel free to work only with classical, i.e. non-derived, schemes here). This implies Grothendieck duality:

Theorem 9.6. *Let X and Y be schemes and $f\colon Y \to X$ a proper smooth map of pure dimension d. Then for every quasicoherent sheaf $\mathcal{M}$ on Y there is a natural isomorphism*

$$f_* \underline{\operatorname{Hom}}_Y(\mathcal{M}, \omega_{Y/X})[d] = \underline{\operatorname{Hom}}_X(f_*\mathcal{M}, \mathcal{O}_X)$$

of (derived) quasicoherent sheaves on X.

If time permits it would be very nice to sketch the following second application of the 6-functor formalism: Preservation of coherent sheaves under (derived) pushforward. What is appealing about this result is that it is very formal and that the proof generalizes easily to other geometric settings like adic spaces and complex analytic spaces (where classical proofs are quite hard). The main challenge in the proof is to find the correct definition of "coherent sheaves" in the first place (i.e. we need a definition for derived sheaves on derived schemes which may not necessarily be noetherian). It turns out that one should work with pseudocompact sheaves:

Definition 9.7. Let A be a discrete ring. An $A_\square$-module $M \in \mathcal{D}_\square(A)$ is called *pseudocompact* if for every uniformly left-bounded family $(N_i)_i \in \mathcal{D}_\square(A)$ we have $\operatorname{Hom}(M, \bigoplus_i N_i) = \bigoplus_i \operatorname{Hom}(M, N_i)$. We denote by $\mathcal{D}_{\square,pc}(A) \subseteq \mathcal{D}_\square(A)$ the full subcategory of pseudocompact objects.

Remark 9.8. If A is a classical ring and $M \in \mathcal{D}_\square(A)$ is discrete then M is pseudocompact if and only if it is pseudocoherent in the sense of [10, §064N]. The "if" direction is easy; here is a sketch of the "only if" direction: Note that if all N_i lie in $\mathcal{D}_\square^{\geq m}(A)$ for some $m \in \mathbb{Z}$ then $\mathrm{Hom}(M, N_i) = \mathrm{Hom}(\tau^{\geq m} M, N_i)$. From this one deduces easily that M is pseudocompact if and only if for all $m \in \mathbb{Z}$, $\tau^{\geq m} M$ is compact as an object in $\mathcal{D}_\square^{\geq m}(A)$. If M is discrete then we can write it as a filtered colimit $M = \varinjlim_i N_i$ of perfect A-modules $N_i \in \mathcal{D}(A)$. Then $\mathrm{Hom}(\tau^{\geq m} M, \tau^{\geq m} M) = \varinjlim_i \mathrm{Hom}(\tau^{\geq m} M, \tau^{\geq m} N_i)$ and by looking at the identity map on the left we deduce that $\tau^{\geq m} M$ is a retract of some $\tau^{\geq m} N_i$, i.e. $\tau^{\geq m} N_i \cong X \oplus \tau^{\geq M}$ for some $X \in \mathcal{D}(A)$. Using [10, Lemma 064X] one concludes that M is $(m-1)$-pseudocoherent. As this is true for all m we see that M is pseudocoherent (see [10, Lemma 064U]).

Remark 9.9. Suppose that A is a noetherian classical ring and $M \in \mathcal{D}_\square(A)$ is discrete and static. Then M is pseudocompact if and only if it is finitely generated. This follows from the previous remark and [10, Lemma 066E].

It is quite formal that pseudocompactness is a Zariski local property (i.e. the assignment $A \mapsto \mathcal{D}_{\square,pc}(A)$ is a Zariski sheaf) – the crucial input is that for an open immersion $\mathrm{Spec}\, B \hookrightarrow \mathrm{Spec}\, A$ of affine schemes the pullback $- \otimes_{A_\square} B_\square$ has bounded Tor dimension (in fact this Tor dimension is bounded by 1). See the second paragraph of the proof of [3, Proposition 9.17] for an argument why finite Tor dimension is enough and see [6, Lemma 2.11.11.(i)] for an argument why the Tor dimension is indeed ≤ 1 (the proof reduces this to the case of standard open immersions, which are easily handled by the above computations).

Definition 9.10. For a scheme X we denote $\mathcal{D}_{\square,pc}(X) \subseteq \mathcal{D}_\square(X)$ the full subcategory of *pseudocompact* sheaves on X. By the above discussion this can be glued out of $\mathcal{D}_{\square,pc}(A)$ for affine open subsets $\mathrm{Spec}\, A \subseteq X$, i.e. a sheaf $\mathcal{M} \in \mathcal{D}_\square(X)$ is pseudocompact if and only if its pullback to every affine subset is pseudocompact in the sense of Definition 9.7.

With the right notion of "coherent" sheaves at hand, it remains to find the correct notion of morphisms along which pseudocompactness is preserved. Classically one usually chooses finite-type proper morphisms of locally noetherian schemes (see [10, Proposition 02O5]), but it is more natural to work with the following generalization:

Definition 9.11. A morphism $f : Y \to X$ of schemes is called *pseudocompact* if locally on X and Y it can be written as a composition of maps $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ such that either $B = A[x]$ or B is pseudocompact as an A-module.

Clearly every perfect morphism of schemes is pseudocompact. In the case of classical (i.e. non-derived) schemes, Definition 9.11 recovers the notion of pseudocoherent morphisms from [10, §067X]. One also sees immediately that every finite-type map of locally noetherian classical schemes is pseudocompact. The following result is formal:

Proposition 9.12. *Let $f : Y \to X$ be a pseudocompact morphism of schemes and $\mathcal{M} \in \mathcal{D}_{\square,pc}(Y)$ a pseudocompact sheaf on Y. Then $f_! \mathcal{M}$ is pseudocompact, i.e. lies in $\mathcal{D}_{\square,pc}(X)$.*

Proof. This can be checked locally on X and Y and is stable under compositions, so we can assume that $X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$ and either $B = A[x]$ or B is pseudocompact as an A-module. In the first case the claim follows easily from adjunctions by using that $f^!$ preserves colimits and is left-bounded by Lemma 9.1 (write $\mathrm{Hom}(f_! \mathcal{M}, \bigoplus_i N_i) = \mathrm{Hom}(\mathcal{M}, f^! \bigoplus_i N_i)$ and continue). If B is pseudocompact as an A-module then f is proper (this can be checked on π_0 and $\pi_0(A) \to \pi_0(B)$ is finite) hence $f_! = f_*$ and therefore $f^! = \underline{\mathrm{Hom}}_A(B, -)$. By pseudocompactness of B one checks that $f^!$ preserves uniformly left-bounded direct sums and is left t-exact, so we can argue as in the previous case. □

Similar to the previous discussion of perfect morphisms, we emphasize that the proof of Proposition 9.12 reduces formally to a *single* computation for the map $\mathrm{Spec}\, A[x] \to \mathrm{Spec}\, A$. We can deduce the following classical result, whose classical proof is much more involved (see e.g. [10, Proposition 02O5]):

Theorem 9.13. *Let $f : Y \to X$ be a proper pseudocompact morphism of schemes. Then for every discrete pseudocompact sheaf $\mathcal{M} \in \mathcal{D}(Y)$, $f_* \mathcal{M}$ is again discrete and pseudocompact.*

Proof. Proper pushforward preserves discreteness (i.e. the usual quasi-coherent sheaves), so this is just a special case of Proposition 9.12. □

Corollary 9.14. *Let $f: Y \to X$ be a proper finite-type map of locally noetherian schemes. Then for every coherent sheaf $\mathcal{M}$ on Y, all $R^i f_* \mathcal{M}$ are coherent sheaves on X.*

Proof. The claim is local on X and Y, so we can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. Then $\mathcal{M}$ is a finite B-module and hence pseudocompact and discrete in $\mathcal{D}_\square(B)$. By Theorem 9.13 $f_* \mathcal{M}$ is pseudocompact and discrete in $\mathcal{D}_\square(A)$ and hence all its cohomology modules are finite A-modules (see [10, Lemma 066E]). □

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