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EUGEN HELLMANN, JOÃO LOURENÇO, LUCAS MANN

1. SUMMARY

The classical Corlette–Simpson correspondence provides a description of representations of the fundamental group of a compact Kähler manifold X (over \mathbb{C}) in terms of so-called Higgs bundles (E, θ) on X , which are pairs consisting of a holomorphic vector bundle E on X and a Higgs field θ , i.e., a map $\theta: E \rightarrow E \otimes \Omega^1$ satisfying $\theta \wedge \theta = 0$. More precisely, there is a canonical equivalence of categories:

$$\left\{ \begin{array}{l} \text{representations } \pi_1(X) \rightarrow \mathrm{GL}(V) \\ \text{on fin. dim. } \mathbb{C}\text{-vector spaces } V \end{array} \right\} \cong \left\{ \begin{array}{l} \text{semistable Higgs bundles on } X \\ \text{with vanishing Chern classes} \end{array} \right\}$$

The p -adic Simpson correspondence (also known as p -adic non-abelian Hodge theory) seeks an analog of this result when X is replaced by a rigid-analytic variety over a p -adic field. This field has a rich history, which we do not want to discuss here, and has gained momentum in recent years due to new available methods in p -adic geometry. In particular, Heuer [6] proved the following analog of the above correspondence:

Theorem 1.1. *Let X be a proper smooth rigid-analytic variety over a complete algebraically closed extension K of \mathbb{Q}_p . Then there is an equivalence of categories*

$$\{v\text{-vector bundles on } X\} \cong \{\text{Higgs bundles on } X\}.$$

Here Higgs bundles are defined very similarly as in the complex case, while v -vector bundles (equivalently pro-étale vector bundles) form a suitable enlargement of the category of representations of the fundamental group. It is still an open question which conditions on a Higgs bundle guarantee that the associated v -vector bundle comes from an actual representation.

There are several variants of the above correspondence. For example, one can replace vector bundles by G -torsors for an algebraic (or even rigid) group G . Also, one can drop the assumption that X is proper, in which case one speaks of the *local p -adic Simpson correspondence*. One can also relax the condition that K is algebraically closed and instead consider finite extensions of \mathbb{Q}_p . In fact, already the case $X = \mathrm{Spa}(K)$ is interesting: Here the two sides of the above correspondence take the following form:

- (a) Let C be the completed algebraic closure of K . By descent, a v -vector bundle on $\mathrm{Spa}(K)$ is the same as a representation of $\mathrm{Gal}(C/K)$ on a finite dimensional C -vector space.
- (b) A Higgs bundle on $\mathrm{Spa}(K)$ is the same as a finite dimensional K -vector space together with an operator. Such a datum is also called a *Sen module*.

Thus the local p -adic Simpson correspondence on $\mathrm{Spa}(K)$ seeks a relation between Galois representations on C -vector spaces and Sen modules. Classically, this relation was studied by Sen [13] in what is now called *Sen theory*; among others

it achieves the construction of a canonical *Sen operator* associated to a suitably nice Galois representation. The recent paper [1] discusses this classical result from a modern geometric perspective by identifying both of the above categories with sheaves on certain stacks. Namely, building on the prismatic site and its cohomology developed by Bhatt–Scholze [5], certain formal algebraic stacks over $\mathrm{Spf}(\mathbb{Z}_p)$ were constructed by Bhatt–Lurie [3]: their formalism provides a notion of Sen operators and Sen modules via coherent sheaves on the Hodge–Tate divisor of the primatization of \mathcal{O}_K . The main result of Anschutz–Heuer–Le Bras in [1] is then the following:

Theorem 1.2. *Let K be a finite extension of \mathbb{Q}_p . The isogeny category of perfect complexes on $\mathcal{O}_K^{\mathrm{HT}}$ embeds fully faithfully into the category of v -vector bundles on $\mathrm{Spa}(K)$ and the essential image consists precisely of “nearly Hodge–Tate” v -vector bundles.*

2. TALKS

Talk 1: Recap on perfectoid spaces.

Recall the notion of perfectoid spaces, e.g. following [9, 1.2, 1.3] and the references therein (the tilting equivalence can be omitted). Show that affinoid perfectoid spaces are sheafy [9, Cor. 1.3.5] and compute the cohomology of the structure sheaf on an affinoid perfectoid [9, 1.3.2.1]. Mention the (far stronger) almost acyclicity [9, 1.3.2.2].

Talk 2: The v -topology.

Introduce the pro-étale site [9, 1.4.1] and in particular the structure sheaf thereon, as well as the v -topology [9, 1.4.2.1], [12, Lecture 17]. Briefly define the notion of a diamond [9, 1.4.2.2], [12, §8.3] (see also [10, §1.4] for a quick overview of the relation between diamonds and rigid varieties). Sketch the proof that v -vector bundles on a perfectoid space are free locally in the analytic/étale topology [12, Lemma 17.1.8]. Show that pro-étale cohomology for certain coverings can be computed in terms of continuous group cohomology [9, 1.4.4.1] in particular Proposition 1.4.38 and Lemma 1.4.39.

Talk 3: \mathbb{G}_a and \mathbb{G}_m in the arithmetic case.

Explain as an example that a v -vector bundle on $X = \mathrm{Spa}(K)$ is the same as a semi-linear Gal_K representation on a \widehat{K} vector space. And that this is the same as a semi-linear Γ -representation on a $\widehat{K_\infty}$ vector space, where K_∞ is the cyclotomic extension. **Do we have references for this?** Then compute the direct images of $R\nu_*\mathcal{O}$, where $\nu : X_{\mathrm{proet}} \rightarrow X$ (see [9, 1.4.4.2] and [14, 3.3, Theorem 1]) and $R\nu_*(\mathcal{O}^*)$ (see [1, Theorem 5.2, ff.]) in the arithmetic setting.

Talk 4: \mathbb{G}_a in the geometric setting.

Compute $R\nu_*\widehat{\mathcal{O}}_X$, where $\nu : X_{\mathrm{proet}} \rightarrow X$, if X is a rigid analytic space over an algebraically closed field. See e.g. [9, 1.4.4, Theorem 1.4.36] and the references therein. Discuss the Hodge–Tate exact sequence

$$0 \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^0(X, \Omega_X(-1)) \rightarrow 0,$$

for proper X as in [11, §3] and [2, §13].

Talk 5: \mathbb{G}_m in the geometric setting.

In the same setting as in Talk 4 compute the cohomology $R\nu_*\widehat{\mathcal{O}}_X$ and discuss the sequence

$$0 \rightarrow \mathrm{Pic}(X) \rightarrow \mathrm{Pic}(X_v) \rightarrow H^0(X, \Omega_X(-1)) \rightarrow 0,$$

(which can be regarded as the p -adic Simpson correspondence for generalized representations in the rank one situation) following [7, Theorem 1.3].

Talk 6: The local correspondence, arithmetic case.

For K a finite extension of \mathbb{Q}_p , the local Simpson correspondence on $X = \mathrm{Spa} K$ encodes classical Sen theory. Explain the main ideas behind Sen theory from [13] and in particular sketch the construction of the Sen operator in [13, Theorem 4]. Follow [1, §1.1] to discuss the subtleties in the correspondence and explain [1, Corollary 5.6] for the rank 1 case. **First formulation of [1, Corollary 5.1] in a more classical setting?**

Talk 7: The local correspondence, geometric case.

Discuss the p -adic Simpson correspondence for small objects when X admits a toric chart $X \rightarrow \mathbb{T}^n$ see [6, §4] and [8, Theorem 6.5] (restricted to $G = \mathrm{GL}_n$).

Talk 8: Prismatic site.

Cover the basics of δ -rings following [5, §2.1-2.2] with emphasis on [5, Definition 2.1, Remark 2.2, and Lemma 2.18]. Introduce the notion and key examples of distinguished elements d of a δ -ring A as in [5, Definition 2.19, Example 2.20] and characterize their Teichmüller series if A is a perfect δ -ring, see [5, Lemma 2.33]. Define prisms (A, I) as in [5, Definition 3.2], prove their rigidity as in [5, Lemma 3.5], and classify perfect prisms via perfectoids, see [5, Theorem 3.10]. Finally, define the prismatic site equipped with a map toward the étale site, see [5, Definition 4.1, Construction 4.4], and state the Hodge–Tate comparison as in [5, Construction 4.9 and Theorem 4.11].

Talk 9: Hodge–Tate stack.

Define the prismaticization of \mathcal{O}_K following [3, Definition 3.1.4], show that it is representably by the stack quotient of a formal affine scheme by an affine group scheme as in [3, Proposition 3.2.3], and prove that it receives a map from $\mathrm{Spf}(A)$ for every \mathcal{O}_K -prism (A, I) . Define the Hodge–Tate divisor $\mathcal{O}_K^{\mathrm{HT}}$ of the prismaticization as in [3, Definition 3.4.1] or [?, Definition 5.1.6], deduce that perfectoid rings R map canonically to $\mathcal{O}_K^{\mathrm{HT}}$, see [3, Example 3.4.3], and identify it with classifying stack of the group G_π in [1, Definition 2.2] following [3, Theorem 3.4.13] and [4, Proposition 9.5]. Since the main source here [3, §3.1-3.4] assumes $K = \mathbb{Q}_p$, the speaker should refer to [1, §2.1] and [4, Example 9.6] for complements on the ramified case.

Talk 10: Prismatic Sen theory.

Define the category $\mathcal{D}(\mathcal{O}_K^{\mathrm{HT}})$ as in [3, Definition 3.5.1], construct the Sen operator Θ following [3, Construction 3.5.4] and compute it for Breuil–Kisin twists $\mathcal{O}_K^{\mathrm{HT}}\{n\}$. Prove [3, Theorem 3.5.8] and [1, Theorem 2.5] describing Hodge–Tate complexes via the Sen operator. Describe the exponential u^Θ for $u \in U_K^1$ as in [3, Proposition 3.7.1] and [1, Lemma 2.14] and reprove Sen’s theorem on continuous semilinear representations of Gal_K following [3, Theorem 3.9.5], compare with [1, §2.3]

Talk 11: Galois cohomology of B_{en} .

Follow [1, §3] to analyze the Hodge–Tate stack of $\mathrm{Spf}(\mathcal{O}_K)$. In particular, define the ring B_{en} and compute its Galois cohomology [1, Theorem 3.12]. This will be crucial for the next talk.

Talk 12: Description of v -vector bundles via Hodge–Tate stacks.

Discuss the comparison between v -vector bundles on $\mathrm{Spa}(K)$ and vector bundles on the Hodge–Tate stack [1, §4]. In particular show the fully faithful embedding of the latter into the former [1, Theorem 4.2] and describe the essential image [1, Lemma 4.6], then explain the application to [1, Corollary 5.1]. If time permits,

explain how the *whole* category of v -vector bundles can be obtained by taking modules on the Hodge–Tate stack for increasing extensions L of K [1, Theorem 4.9].

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